Laser Acceleration of Particles with the Plasma Vector-Soliton

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Dedicated to Professor Dieter Pfirsch on his 60th Birthday

The closed, energy conserving system of electromagnetic-plasma equations describing the classical beat wave accelerator are shown to lead to an intrinsic limit on the acceleration efficiency due to the pump depletion effect producing an intense trail of plasma waves. We propose and analyze with theory and simulations a wakeless accelerating structure composed of a nonlinear vector soliton. The solitary wave speed u of the vector soliton can be greater than the group velocity of the electromagnetic pulse as determined by the laser intensity $a_0 = e\,E_0/m\,\omega_0\,c$ through the soliton dispersion relation. Combining the triple soliton structure with a weak plasma density depression of radius $(c/\omega_p)\,(\omega_0/\omega_p)^{1/2}$, we compute an accelerating structure with $E_p \lesssim m_e\,\omega_p\,c/e$ moving at the speed of light and limited in length only by the conditions set by beam loading or plasma nonuniformities.

1. Principles of Laser Acceleration of Particles

The largest acceleration produced in a laboratory on a macroscopic scale is that of an electron in a high intensity laser beam. The strength of the acceleration is measured by the quiver velocity, the velocity produced by the acceleration in one wave period, compared with the speed of light. This dimensionless ratio defined by $a = eE/m\,\omega\,c$ is a Lorentz scalar given by the invariant length of the four vector potential A^μ. For $a = (e/m\,c)\,(-A_\mu A^\mu)^{1/2} \sim 1$ the electric potential across the wavelength $e\,E\,(c/\omega)$ equals the rest mass energy of the electron, and the electron quiver velocity is close to the speed of light.

The basic problem of laser acceleration is to find an efficient mechanism to transform the transverse electric field of the laser into a longitudinal accelerating structure. The second requirement for high energy acceleration is to match the phase velocity of the accelerating structure to the particle velocity over the distance required to obtain the final particle energy. After these mechanisms are found, the third question is the issue of efficiency. There are three recent approaches to producing an intense accelerating structure from the transverse laser field.

1. The use of microscopic structures such as liquid metal droplets [1] or the crystal structure [2] of solids to form a near field accelerating structure. These may be called mini-linacs. For the droplets the accelerating potential is limited to about 1 GeV/m by the breakdown of the droplets into a plasma. For the use of crystals with X-ray wave-length electromagnetic fields the damage of the material and the beam interaction with the matter are going to limit the physically attainable values which are still under investigation.

2. The use of a wiggler or undulator magnetic field [3, 4] to produce a beat wave accelerating structure with variable parameters. Without a plasma component this accelerator is the inverse free electron laser (IFEL) in which the ponderomotive buckets are used to trap and accelerate particle bunches. Typical estimates for the accelerating field are 0.2 GeV/m. The maximum electron energy produced by the IFEL is thought to be limited by the intense synchrotron radiation losses to less than 1 TeV.

With a plasma component [5] present, the strength of the acceleration may be increased up to 1 GeV/m but the synchrotron losses still pose a limit to the maximum acceleration energy.

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3. The plasma beat wave acceleration scheme [6–11] is a natural way to achieve very intense acceleration fields without regard to material breakdown and should be free of many troublesome laser-plasma instabilities if the laser pulse length is sufficiently short. In this method the longitudinal acceleration arises from the beat Lorentz force $e(\mathbf{v}_1 \times \mathbf{B}_2 + \mathbf{v}_2 \times \mathbf{B}_1)$ driving a plasma wave at resonance $\omega_p = \omega_2 - \omega_1 = c(k_2 - k_1)$ nearly to the wave breaking amplitude $E_{max} = cm\,\omega_P/e$. Coherent accelerating fields of order 1–10 GeV/m appear feasible with this plasma-laser acceleration scheme.

The principle accelerator issues for the beat plasma wave scheme are the extent or distance over which the matching of the beat wave to the speed $v = c(1 - 1/2\gamma^2)$ of the ultrarelativistic particle $\gamma = E/mc^2 \gg 1$ can be achieved, and the efficiency of the acceleration.

The efficiency question has two fundamental aspects. The first part of which is the efficiency [11, 12] of the growth of the plasma wave, and the second is the degree of beam loading of the accelerated particles.

The importance of and the severity of the condition on efficient acceleration in high energy acceleration may be appreciated by the following consideration based on the required luminosity $\mathscr{L}$ and the asymptotic form of the high energy scattering cross-section. Since the relevant cross-section is given by $\sigma \approx \pi(\hbar/p)^2 \cong (\hbar c/E_{cm})^2$, decreasing inversely proportional to the square center-of-mass energy E_{cm}, the luminosity $\mathscr{L}$ of the beam has to increase as E_{cm}^2 in order to keep the number of relevant events in a given time, $R = \sigma\mathscr{L}$, constant. Since the luminosity is given by $\mathscr{L} = N^2 f/\pi a^2$ and the beam power by $P_b = E_{cm} N f$, where N is the particle number in a bunch, f is the bunch frequency and a is a beam radius, the necessary power supplied to the beam increases as $P_b = \alpha(Rf)^{1/2} \cdot (a/a_b)(E_{cm}^2/13.6\text{ eV})$. For an example, at $E_{cm} =$ 10 TeV for colliding e^+e^-, the required luminosity is $\mathscr{L} = 10^{34}\,\text{cm}^{-2}\,\text{sec}^{-1}$ and P_b is several GW. Then, the necessary laser power supplied to an accelerator is $P_{laser} = P_b/\eta_c$ where η_c is the coupling efficiency between laser and particle beams. Since P_b is already required to be a very high power, the coupling efficiency is an important issue in order to keep the laser power within a realistic level.

In this work we discuss our results for the method of producing a phased locked plasma wave soliton [13] from the relativistic three wave interactions involving two transverse electromagnetic waves and a longitudinal plasma wave. To the extent that the speed of this relativistic plasma wave soliton can be tuned to the speed of the ultrarelativistic particles, both the phase matching condition and the efficiency of the growth and maintenance of the plasma wave questions are answered.

In Sect. 2 we describe the beat Lorentz force mechanism for converting the transverse electromagnetic fields into longitudinal accelerating fields by analyzing the single particle Hamiltonian. Sect. 3 we analyze the pump depletion problem from the energy conserving coupled fields equations. In Sect. 4 we give the solution to the pump depletion problem in terms of the wakeless triple soliton theory for the coupled fields. In Sect. 5 we conclude with a discussion of the properties of the relativistic soliton solutions where we attempt to explain the meaning of those subclass of solutions that contain a phase flip point propagating faster than the speed of light.

2. Beat Wave Acceleration from Transverse Electromagnetic Waves

The possibility of transforming the transverse electric field into a longitudinal accelerating structure is most easily seen from the relativistic single particle Hamiltonian $H = H(\mathbf{P}, \mathbf{A}(x, t))$. The vector potential for two waves is

$$\mathbf{A} = \hat{\mathbf{e}}_y A_0 \cos(k_0 x - \omega_0 t + \psi_0) + \hat{\mathbf{e}}_y A_1 \cos(k_1 x - \omega_1 t + \psi_1), \quad (1)$$

from which the electric and magnetic field strengths are $E_j = \omega_j A_j/c$ and $B_j = k_j A_j$. For the case in which the laboratory is the rest-frame of the first wave ($\omega_0 = 0$) the field $E_0 = 0$, $B_0 = k_0 A_0 = k_w A_w$ is the magnetic field of the magnetic wiggler or undulator.

The dimensionless field strengths are

$$a_0 = eA_0/mc^2, \quad a_1 = eA_1/mc^2. \quad (2)$$

For a strong wiggler the $a_w \sim 1$, while for a focused high power laser pulse a_j is of order a few tenths.

The relativistic motion of the single particle is given by

$$H^2 = m^2c^4 + c^2(p_x^2 + p_z^2) + (cP_y - eA_y(x, t))^2, \quad (3)$$

where $p_{x,z} = m\gamma v_{x,z}$ and $P_y = m\gamma v_y + (e/c)A_y(x,t)$. In the one-dimensional limit the Hamiltonian has the symmetry $\partial H/\partial y = \partial H/\partial z = 0$ so that P_y and P_z are constants of the motion. Thus, the motion $H(p_x, x, t)$ takes places in a $d = 3$ phase space often described as a $1\frac{1}{2}-D$ dimensional system.

In the one-wave limit the system is integrable and results in "figure eight" orbits of the electrons given by $x\,\omega/c = \frac{1}{8}(a/\gamma)^2 \sin 2\eta$, $y\,\omega/c = (a/\gamma)\cos\eta$ with $\gamma^2 = (E/mc)^2 = 1 + \frac{1}{2}a^2$ and $\eta = k\cdot x \equiv \omega t - kx$. For two waves there is the superposition of the two figure eight orbits plus the nonlinear interactions.

In the presence of two waves the nonlinear motion with constant P_y, P_z is given by

$$\begin{aligned}&H^2 - m^2c^4 - c^2(P_y^2 + P_z^2)\\ &= c^2p_x^2 - 2ecP_y(A_0\cos k_0\cdot x + A_1\cos k_1\cdot x)\\ &\quad + e^2A_0^2\cos^2(k_0\cdot x) + e^2A_1^2\cos^2(k_1\cdot x)\\ &\quad + e^2A_0A_1[\cos(k_0+k_1)\cdot x + \cos(k_0-k_1)\cdot x],\end{aligned}$$

where $k\cdot x \equiv \omega t - kx$. The new drifts of the oscillation centers of the hight frequency motion are produced by the beat interaction potentials

$$A_0A_1\cos[(k_0 \pm k_1)\cdot x].$$

Averaging over the high frequency oscillations the drift motion in the slow ponderomotive potential is given by the interaction Hamiltonian

$$h(x,p_x,t) = mc^2\left[1 + \frac{p_x^2}{m^2c^2} + \tfrac{1}{2}a_0^2 + \tfrac{1}{2}a_1^2 + a_0a_1\cos(k_0\pm k_1)\cdot x\right]^{1/2} \quad (4)$$

with $p_y = p_z = 0$. The equations of motion are

$$\dot{x} = \frac{\partial h(x,p_x,t)}{\partial p_x}, \quad \dot{p}_x = -\frac{\partial h(x,p_x,t)}{\partial x} \quad (5)$$

and $\gamma v_y = -(e/mc)A_y(x,t) + P_y/m$.

For two waves with $\omega_0, \omega_1 > 0$ the acceleration is produced by the beat ponderomotive potential in (4) traveling with the speed

$$v_{\rm ph} = \frac{\omega_0 - \omega_1}{k_0 - k_1} \cong \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega_0^2}\right)^{1/2}, \quad (6)$$

and the associated relativistic energy of a trapped particle is

$$\gamma = (1 - v_{\rm ph}^2/c^2)^{1/2} = \omega_0/\omega_p. \quad (7)$$

In deriving (6) we use the transverse plasma wave dispersion relation

$$D(k) = -k\cdot k + \frac{\omega_p^2}{c^2} = k^2 + \frac{\omega_p^2 - \omega^2}{c^2} = 0. \quad (8)$$

For such transverse waves the Lorentz invariant plasma frequency $\omega_p = (4\pi n_e e^2/m)^{1/2}$ plays the role of an effective photon mass. The phase velocity ω/k and group velocity $v_g = d\omega/dk$ are related by $v_g(\omega/k) = c^2$ with $\omega/kc = (1 + \omega_p^2/k^2c^2)^{1/2} > 1$ and $v_g \lesssim c$ for $\gamma = \omega_0/\omega_p \gg 1$. Since $\omega/kc > 1$ there is no Landau damping of the tranverse plasma wave.

The acceleration of a particle by the beat wave Hamiltonian (4) is given by

$$\begin{aligned}\dot{p}_x &= \frac{mc^2k_b a_0a_1\sin[k_0(x - v_{\rm ph}t)]}{2[1 + \frac{1}{2}a_0^2 + \frac{1}{2}a_1^2 + p_x^2/m^2c^2]^{1/2}}\\ &= eE_{\rm eff}\sin[\Delta k(x - v_{\rm ph}t)] \quad (9)\end{aligned}$$

and

$$\dot{x} = \frac{p_x}{m[1 + \frac{1}{2}a_0^2 + \frac{1}{2}a_1^2 + p_x^2/m^2c^2]^{1/2}}, \quad (10)$$

where we define the beat wave number and frequency $k_b = k_0 - k_1$ and $\omega_b = \omega_0 - \omega_1$. For $c(k_0 - k_1) = ck_b \cong \omega_p$ the strength of the effective electric field is $E_{\rm eff} = (m_c\omega_p/e)(a_0a_1/2\gamma)$. We define $\gamma = \gamma_\perp\gamma_\parallel(\dot{x})$ where $\gamma_\perp = (1 + \frac{1}{2}a_0^2 + \frac{1}{2}a_1^2)^{1/2}$ and $\gamma_\parallel(\dot{x}) = (1 - \dot{x}^2/c^2)^{1/2}$.

In the presence of a plasma the electric field $E_{\rm eff}$ gives rise to a displacement $\xi(x,t)$ of electrons initially at x away from the ions which produces the collective longitudinal electric field $E_x = 4\pi ne\xi(x,t)$. Including the self-consistent plasma eE_x electric field and calculating the acceleration $p_x = m\gamma_\perp\gamma_\parallel(\dot{\xi})\dot{\xi}$ of the plasma electrons from (9)–(10) yields

$$\gamma_\perp\gamma_\parallel^3(\dot{\xi})\ddot{\xi} + \omega_p^2\xi = \left(\frac{e}{m}\right)E_{\rm eff}\sin[k_b(x - v_{\rm ph}t)] \quad (11)$$

for the collective plasma wave driven by the ponderomotive beat electric field. For the beat frequency $\omega_b = k_b v_{\rm ph} \cong \omega_p$ the plasma frequency the amplitude of the electric field grows secularly until the relativistic detuning from $\gamma_\parallel^3(\dot{\xi})$ takes the plasma wave out of resonance. Following the theory of Rosenbluth and Liu [14] yields the time $t_m\omega_p = c_1(a_0a_1)^{-2/3}$ to reach the maximum field $E_m = c_2(mc\omega_p/e)(a_0a_1)^{1/3}$ as computed in detail in Tang et al. [15] and Horton and Tajima [12]. Off resonance the collective electric field is given by $E_x =$

$4\pi n e \xi \cong [\omega_p^2/(\omega_p^2 - \gamma_\|^3 \omega_b^2)] E_{\text{eff}}$ with $\omega_b = \omega_0 - \omega_1$.

In the presence of a wiggler field $\omega_0 = 0$, $a_w = e A_0/m c^2 = e B_w/m c^2 k_w$ the accelerating potential is $e^2 A_0 A_1 \cos[(k_w + k_1) x - \omega_1 t]$ with the phase velocity

$$v_{\text{ph}} = \frac{\omega_1}{k_1 + k_w} \cong c\left(1 - \frac{k_w}{k_1} + \frac{\omega_p^2}{2 k_1^2 c^2}\right). \quad (12)$$

The energy of a trapped particle is given by

$$\gamma^2 = \frac{1}{2 k_w/k_1 - \omega_p^2/k_1^2 c^2}.$$

From (4) and (5) the acceleration is given by

$$\dot{p}_x = \frac{m c^2 (k_0 + k_1) a_w a_1 \sin[(k_w + k_1)(x - v_{\text{ph}} t) + \Delta\psi]}{2[1 + \frac{1}{2} a_w^2 + p_x^2/m^2 c^2]^{1/2}}$$
$$= e E_{\text{eff}} \sin[(k_w + k_1)(x - v_p t) + \Delta\psi] \quad (13)$$

and

$$\dot{x} = \frac{p_x/m}{(1 + \frac{1}{2} a_w^2 + p_x^2/m^2 c^2)^{1/2}}.$$

Now the parameters a_w and k_w can be made slowly vaying functions of x to allow the trapping potential to change shape and speed. The amplitude of the accelerating field in the inverse FEL is $E_{\text{eff}} = (m c \omega_1/e)(a_w a_1/2\gamma)$.

For both the beat wave accelerator (6)–(11) and the inverse FEL (12)–(13) the trapping of the resonant particles $\gamma \cong \gamma_r$ is analyzed by transforming to the Lorentz phase variable $\theta = (k_0 \pm k_1) \cdot x$ and expanding the Hamiltonian in p_x about the resonant momentum $\delta p = (p_x - p_r)/mc$, where $d_t \theta = 0$ at $p_x = p_r$. Using (9) or (13) the particle motion near the resonance is given by

$$\frac{d\theta}{dt} = (k_0 \pm k_1) \left.\frac{\partial^2 h}{\partial p_x^2}\right|_{p_r} \delta p = \frac{c (k_0 \pm k_1)\, \delta p}{\gamma_r^3},$$
$$\delta\dot{p} = (e E_{\text{eff}}/m c) \sin\theta, \quad (14)$$

where $e E_{\text{eff}}/m c = \omega_p a_0 a_1/2\gamma_r$ for the beat wave accelerator and $e E_{\text{eff}}/m c = \omega_1 a_w a_1/2\gamma_r$ for the inverse FEL. In both cases trapped particle oscillations occur in the beat wave potential buckets with frequency $\omega_\beta^2 = c^2 (k_0 \pm k_1)^2 a_0 a_1/2\gamma_r^4$.

Now we consider the reaction of the electrons on the electromagnetic fields. In the limit of low plasma density the coupling of A_0, k_0 and A_1, k_1 by the electrons may be viewed as Compton scattering from free electrons where the acceleration $\dot{p}_x$ is the recoil from the wave scattering. In the regime of higher electron density it is the plasma wave that absorbs the recoil of the scattered electromagnetic wave. This is called the Raman regime. In this regime the strength of the accelerating field increases with the square root of the plasma density.

3. Coupled Field Equations and Pump Depletion

The electromagnetic fields in (1) couple to the plasma component through the generation of a longitudinal accelerating field $E_x = -\partial_x \Phi(x, t)$. The field equations for the system are $\partial_\mu F^{\mu\nu} = 4\pi J^\nu/c$ with $\nu = 0, 2$ being nontrivial. For $\nu = 2(y)$ we calculate the current by $J_y = \sum e n v_y = -(n_e e^2/m c) A_y$ using the conserved P_y of Section 2. Separating the density $n_e = n_0 + \delta n_p$ into its average $n_0 = z n_i$ and fluctuating components, we obtain for the $\nu = 2$ and 0 field equations

$$\left(\partial_\mu \partial^\mu + \frac{\omega_p^2}{c^2}\right) A_y(x, t) = \frac{4\pi e\, \delta n_p}{c} A_y, \quad (15)$$

$$\nabla^2 \Phi(x, t) = -4\pi e\, \delta n_p. \quad (16)$$

In the present work we neglect backscattering and assume that $k_1, \omega_1 \gg k_p, \omega_p$ to reduce (15) to an envelope equation for the amplitude $a_j(x, t)$ and phase $\psi_j(x, t)$ of the transverse fields. Using

$$D(k^\mu = k_j^\mu - i\,\partial^\mu) = D(k_j) - i\,(\partial D/\partial k^\mu)\,\partial^\mu$$

with $D(k_j) = 0$ we obtain the reduced mode coupling equations

$$(\partial_t + v_g \partial_x)\, a_0 = -\frac{e}{2m\omega_0} \frac{\partial^2 \Phi}{\partial x^2} a_1 \sin(\vartheta + \psi_t), \quad (17)$$

$$(\partial_t + v_g \partial_x)\, a_1 = \frac{e}{2m\omega_1} \frac{\partial^2 \Phi}{\partial x^2} a_0 \sin(\vartheta + \psi_t), \quad (18)$$

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) \psi_t$$
$$= \frac{e}{2m} \frac{\partial^2 \Phi}{\partial x^2} \left(\frac{a_0}{\omega_1 a_1} - \frac{a_1}{\omega_0 a_0}\right) \cos(\vartheta + \psi_t), \quad (19)$$

where $\psi_t = \psi_0 - \psi_1$,

$\vartheta = \vartheta_0 - \vartheta_1 = k_b x - \omega_b t$ and $\vartheta_j = k_j x - \omega_j t$.

In (17)–(19), we neglected the difference of the group velocities of the two electromagnetic waves which is $v_0 - v_1 = c/\gamma_0^3$. Truncating the electromagnetic modes to (17)–(19) neglects coupling to multiple Raman side bands [13].

The ponderomotive force on the electrons $f_d = -e\left\langle v_y \frac{\partial A_y}{\partial x}\right\rangle c$ averaged over $\vartheta_0 + \vartheta_1$ at fixed $\vartheta_0 - \vartheta_1$ gives the equation for the longitudinal electron fluid motion,

$$\frac{\mathrm{d}\gamma_v v_x}{\mathrm{d}t} = \gamma_v^3 \frac{\mathrm{d}v_x}{\mathrm{d}t} \tag{20}$$

$$= \frac{e}{m}\frac{\partial \Phi}{\partial x} + \frac{(k_1 - k_2)\, c^2}{2} a_1 a_2 \sin(\vartheta + \psi_t),$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x}$ is the Lagrangian derivative and $\gamma_v = (1 - v_v^2/c^2)^{-1/2}$. We note that (20) for the cold electron fluid is the same as (9) for the single particle acceleration accept for the presence of the collective field $(e/m)\,\partial_x \Phi$ and the approximation $\gamma_\perp = (1 + \frac{1}{2} a_b^2 + \frac{1}{2} a_1^2)^{1/2} \cong 1$.

To reduce (20) we introduce the Lagrangian coordinate x_L and displacement $\xi(x_L, t)$ with $v_x = \mathrm{d}\xi/\mathrm{d}t$ and

$$x_L = x - \xi(x, t). \tag{21}$$

Note there that $\mathrm{d}/\mathrm{d}t = (\partial/\partial t)_{x_L}$. Combining the Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi e (n - n_0)$$

and the charge conservation law (or Ampére's law)

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial x}\right) + 4\pi e n v_x = 0$$

gives the total derivative of E_x by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \Phi}{\partial x} = -4\pi e n_0 v_x,$$

with integrates to give

$$\frac{\partial \Phi}{\partial x} = -4\pi e n_0 \xi, \tag{22}$$

where n_0 is the homogeneous ion density. We define the dimensionless plasma wave field by

$$a(x, t) = \frac{-e}{m\,\omega_\mathrm{p} c}\frac{\partial \Phi}{\partial x} = \frac{\omega_\mathrm{p}\,\xi(x, t)}{c}. \tag{23}$$

The Eqs. (20), (22), and (23) yield

$$\gamma_{\dot a}^3 \ddot a + \omega_\mathrm{p}^2 a = \frac{c}{2} k_b c\, \omega_\mathrm{p} a_0 a_1 \sin(\vartheta + \psi_t), \tag{24}$$

where

$$\gamma_{\dot a} = [1 - (\dot a/\omega_\mathrm{p})^2]^{-1/2},$$

and

$$\dot a = \frac{\mathrm{d}a}{\mathrm{d}t} \quad \text{and} \quad \ddot a = \frac{\mathrm{d}^2 a}{\mathrm{d}t^2}.$$

The energy density in the longitudinal plasma wave is

$$W_\mathrm{p} = n_e m c^2 (1 - \xi^2/c^2)^{-1/2} + \frac{1}{8\pi}\left(\frac{\partial \Phi}{\partial x}\right)^2, \tag{25}$$

where $n_e m c^2 \gamma$ is the electron energy density and $(\partial_x \Phi)^2/8\pi$ is the plasma wave electric energy density, respectively. Normalizing W_p by nmc^2, (25) yields

$$\frac{W_\mathrm{p}}{mnc^2} = \tilde W_\mathrm{p} = \tfrac{1}{2} a^2 + [1 - (\dot a/\omega_\mathrm{p})^2]^{-1/2}. \tag{26}$$

In terms of a_j, the total energy density of the electromagnetic waves is

$$\frac{W_t}{mnc^2} = \tilde W_t = \frac{1}{2}\left[\left(\frac{\omega_0}{\omega_\mathrm{p}}\right)^2 a_0^2 + \left(\frac{\omega_1}{\omega_\mathrm{p}}\right)^2 a_1^2\right].$$

Now we derive the energy conservation law. Differentiating (26) with (24) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde W_\mathrm{p} = \frac{1}{2}\frac{k_b c}{\omega_\mathrm{p}} a_0 a_1 \dot a \sin(\vartheta + \psi_t). \tag{27}$$

The Eqs. (17) and (18) conserve the electromagnetic action density $\omega_0 a_0^2 + \omega_1 a_1^2$ and give the following energy equation

$$\frac{\partial \tilde W_t}{\partial t} + \frac{\partial}{\partial x}(v_\mathrm{g} \tilde W_t) = \frac{c}{2}\frac{\omega_b}{\omega_\mathrm{p}}\frac{\partial a}{\partial x} a_1 a_0 \sin(\vartheta + \psi_t). \tag{28}$$

When a is only a function of ϑ, then

$$\partial_t a + \frac{\omega_b}{k_b}\partial_x a = 0.$$

Therefore, (27) and (28) yield the total power balance equation

$$\frac{\partial}{\partial t}(W_t + W_p) + \frac{\partial}{\partial x}(v_g W_t) = 0. \qquad (29)$$

An exact law for the decay of the total transverse electromagnetic energy $\varepsilon = \int dx' \, W_t(x', t)$ follows from the integral of (29) over the electromagnetic wave pulse

$$\frac{d\varepsilon_t}{dt} = \int_{-L_p}^{0} dx' \frac{d}{dt} W_t(x', t) \cong - \int_{-L_p}^{0} dx' \left(\frac{\partial W_p}{\partial t}\right)_x .$$

Here $x' = x - v_g t$ is the wave frame coordinate and the electromagnetic pulse length is L_p. Since

$$\left(\frac{\partial W_p}{\partial t}\right)_x = \left(\frac{\partial W_p}{\partial t}\right)_{x'} - v_g \frac{\partial}{\partial x'} W_p$$

and since

$$\left| v_g \frac{\partial}{\partial x'} W_p \right| \cong \frac{v_g}{L_p} W_p \gg \left| \left(\frac{\partial W_p}{\partial T}\right)_{x'} \right| \cong \frac{v_g W_p}{L_p W_t} W_p ,$$

we have

$$\frac{d\varepsilon_t}{dt} \cong - v_g W_p(x' = -L_p), \qquad (30)$$

which states that the decay of transverse energy is from the plasma wave energy flux at the tail of the electromagnetic pulse.

For constant a_0 and a_1, the plasma wave amplitude satisfying (24) oscillates periodically due to the relativistic detuning [14, 15] of the plasma oscillator from the $\gamma_{\dot{a}}$ dependence of the electron mass. The maximum of the plasma wave amplitude a_m has the value

$$\langle a_m^2 \rangle = (16/3)^{2/3} (a_0 a_1)^{2/3} \qquad (31)$$

at $\vartheta_m = \omega_b t_m (\text{or } k_b x'_m) \cong 5.6\,(16/3)^{1/3} (1/a_0 a_1)^{2/3}$. We define the pump depletion time τ_{pd} as the time at which the electromagnetic pulse has lost one-half its initial energy and use a_{pm} for W_p. The equations (30) and (31) yield the pump depletion time,

$$\tau_{pd} = \left(\frac{3}{4}\right)^{2/3} \frac{L_p}{c} \left(\frac{\omega_0}{\omega_p}\right)^2 \left(\frac{a_0 a_1}{4}\right)^{1/3} . \qquad (32)$$

Although the pump depletion time increases with the laser pulse length, the relativistic nonlinearity coupled with the amplitude oscillation of the electromagnetic wave modulates the plasma wave phase,

$$\psi_p(x, t) = \int_0^t \Delta\omega \, dt' = \int_0^t \tfrac{3}{8} \langle a^2(x, t') \rangle \, \omega_p \, dt'$$

in the tail of the pulse, which leads to the dephasing between particles and acceleration fields. Therefore, L_p should be limited by $\psi_p(x, \tau_{mod}) = \pi$. Namely,

$$L_{p\,max} \cong \frac{8\pi}{3} \frac{1}{\langle a^2 \rangle} \frac{c}{\omega_p} . \qquad (33)$$

The maximum amplitude $\langle a_m^2 \rangle = \frac{1}{2} \left(\frac{16}{3} a_0 a_1\right)^{2/3}$ and (33) yield

$$L_{p\,max} \cong \pi \left(\frac{16}{3}\right)^{1/3} \frac{c}{\omega_p} \left(\frac{1}{a_0 a_1}\right)^{2/3} , \qquad (34)$$

which limits the acceleration length as

$$c\,\tau_{pd} \cong \pi \left(\frac{3}{16 a_0 a_1}\right)^{1/3} \left(\frac{\omega_0}{\omega_p}\right)^2 \frac{c}{\omega_p} . \qquad (35)$$

from the pump depletion due to emission of plasma waves.

The above difficulties of the limitation of the acceleration length and the inefficient acceleration due to the energy loss by the wake forces us to introduce the following wakeless vector soliton concept.

4. Envelope Equation and Wakeless Triple Soliton

When electromagnetic pulses are not too strong, the driven plasma wave is described by

$$a(x, t) = a_p(x, t) \cos(\vartheta + \psi_p) . \qquad (36)$$

The envelope approximation for a from (24) and (17)–(19) yields

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) a_0 = - \frac{\omega_p k_b c}{4 \omega_0} a_1 a_p \cos(\psi_t - \psi_p) ,$$

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) a_1 = \frac{\omega_p k_b c}{4 \omega_1} a_0 a_p \cos(\psi_t - \psi_p) ,$$

$$\frac{\partial}{\partial t} a_p = \frac{\omega_p k_b c}{4 \omega_b} a_0 a_1 \cos(\psi_t - \psi_p) ,$$

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) \psi_t \qquad (37)$$

$$= - \frac{\omega_b k_b c}{4} a_p \left(\frac{a_0}{\omega_1 a_1} - \frac{a_1}{\omega_0 a_0}\right) \sin(\psi_t - \psi_p) ,$$

$$\frac{\partial}{\partial t} \psi_p = \left(\Delta\omega - \frac{3}{16} \omega_p a_p^2\right) + \frac{k_b c \,\omega_p}{4} \frac{a_0 a_1}{\omega_b a_p} \sin(\psi_t - \psi_p),$$

where $\Delta\omega = \omega_b - \omega_p$. If we introduce the complex amplitude functions

$$\phi_0 = \tfrac{1}{2} a_0 e^{i\psi_0}, \quad \phi_1 = \tfrac{1}{2} a_1 e^{i\psi_1} \quad \text{and} \quad \phi_p = \frac{a_p}{2} e^{i\psi_p},$$

(37) yields

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right)\phi_0 = \beta_0 \phi_1 \phi_p,$$

$$\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right)\phi_1 = -\beta_1 \phi_0 \phi_p^*, \tag{38}$$

$$\left[\frac{\partial}{\partial t} - i\Delta\omega - \frac{3}{4} i\omega_p |\phi_p|^2\right]\phi_p = -\beta_p \phi_0 \phi_1^*,$$

where $\beta_0 = \omega_p k_b c/2\omega_0$, $\beta_1 = \omega_p k_b c/2\omega_1$ and $\beta_p = \omega_p k_b c/2\omega_b$.

Given the initial electromagnetic pulses ϕ_0 and ϕ_1, (37) or (38) are advanced in space-time to compute the growth of $\phi_p(x,t)$ and the subsequent reaction on ϕ_0, ϕ_1. In the simplest approximation the laser is taken with fixed or frozen amplitudes and phases. The plasma wave grows initially as $a_p \cong \omega_p t a_0 a_1/4$. reaches the maximum $a_p^m = (16 a_0 a_1/3)^{1/3}$ where ψ_p rotates from ψ_t to $\psi_t + \pi$ and whereupon a_p decays to start the cycle over again. These limit cycle oscillations are repeated indefinitely leaving a train of plasma waves that rapidly deplete the laser pulse [12].

For square electromagnetic laser pulses we have shown [12] that the most coherent pulse arises when the laser pulse length is chosen to equal that of the cycle time (34) for the plasma wave to return to zero amplitude. This choice eliminates the initial wave according to (30). After a time of order $t_p \sim \omega_p^{-1}(\omega_0/\omega_p)^2$, however, the pulse strongly changes shape with the energy compressed to the rear of the pulse. In the reshaped pulse the plasma wave emission now becomes strong again [14].

Recently, we have shown [13] that by introducing a phase flip $\psi_t \to \psi_t \pm \pi$ in one of the electromagnetic waves so that $\phi_0 \phi_1$ becomes an odd function about the position of the phase flip. the plasma wave can be reabsorbed in the second half of the pulse. In this manner a wakeless electromagnetic-plasma wave pulse can be constructed. Such pulses were recently simulated by numerical solutions of (38) and by particle simulations. Here we show the possibility of such localized, one-dimensional solutions by finding the interaction Hamiltonian $H(a_j, \psi_j)$ for solutions of the form $x' = x - ut$.

For solutions of (38) of the form $f(x-ut)$ we have $(\partial_t + v_g \partial_x) = (v_g - u)\partial_{x'}$ and it is easy to see that the action I_j invariants are

$$I_0 + I_1 = \text{const}, \qquad I_0 + I_p = \text{const}, \tag{39}$$

where

$$I_{0,1} = \omega_{0,1}(v - v_g) a_{0,1}^2 \quad \text{and} \quad I_p = \omega_p a_p^2.$$

In terms of the action I_j and angle ψ_j ($j = 0, 1, p$) variables the one-dimensional solutions of (38) are given by the interaction Hamiltonian

$$H(I,\psi) = \left(\Delta\omega + \frac{3}{32}\frac{I_p}{u}\right)\left(\frac{I_p}{u}\right) + \frac{2(I_0 I_1 I_p)^{1/2}\sin(\psi_0 - \psi_1 - \psi_p)}{[\omega_0 \omega_1 \omega_p (u - v_g)^2 u]^{1/2}}, \tag{40}$$

where

$$\frac{dI_j}{dx'} = -\frac{\partial H}{\partial \psi_j}, \quad \frac{d\psi_j}{dx'} = \frac{\partial H}{\partial I_j}. \tag{41}$$

The 2D system $(I_0, I_1, I_p, \psi = \psi_0 - \psi_1 - \psi_p)$ is integrable since $I_0 + I_1$, $I_0 + I_p$ and H are constant of the motion.

For a localized plasma wave $I_p(x' = \pm\infty) = 0$, we have $H = 0$. Solving for $\sin\psi$ from $H = 0$ allows one to compute $a_0 a_1 \cos(\psi_t - \psi_p)$ for the growth and decay of the plasma wave da_p/dx'. We obtain

$$\frac{\partial a_p}{\partial t} = \pm \frac{\omega_p}{4}\left[a_0^2(x')\, a_1^2(x') - 4\left(\frac{\Delta\omega}{\omega_p} + \frac{3}{32} a_p^2\right)^2 a_p^2\right]^{1/2}, \tag{42}$$

where $a_0(x')$ and $a_1(x')$ follow from the action integrals (39) and the initial data

$$a_0^2(x') = a_0^2(x_0{}') - \left(\frac{u}{u - v_g}\right)\left(\frac{\omega_p}{\omega_0}\right) a_p^2(x'), \tag{43}$$

$$a_1^2(x) = a_1^2(x_0') + \left(\frac{\omega_0}{\omega_1}\right)[a_0^2(x_0') - q_0^2(x')].$$

The branch ($\pm$) of (42) changes from growth (+) in the head of the laser pulse to decay (−) at the point of the phase flip $\psi_t \to \psi_t \pm \pi$.

A simple, analytic solution of (38) or (42) occurs when the (3/16) a_p^2 frequency shift term is dropped. Taking ϕ_0 to have the phase flip the solution is $\phi_0 = -\hat{a}_0 \tanh(qx')$, $\phi_1 = \hat{a}_1 \operatorname{sech}(qx')$ and $\phi_p = \hat{a}_p \operatorname{sech}(qx')$ where $p = (\hat{a}/2)(\omega_p/\omega_0)^{1/2}\omega_p/[u(u - v_g)]^{1/2}$ and $u = v_g/(1 - \omega_p \hat{a}_p^2/\omega_0 \hat{a}^2)$. The velocity u can be made equal to the speed of light when $\hat{a}_p^2/\hat{a}^2 = \omega_p/2\omega_0$ where we use $v_g \cong c(1-\omega_p^2/2\omega_0^2)$. Qualitatively similar solutions occur for the full equations (37) which contain the parameters $\Delta\omega/\omega_p$ as shown by (40)−(43). The optimal choice for $\Delta\omega$ is discussed in Mima et al. [13].

In conclusion, we have developed the theory of the pump depletion effect on the relativistic plasma beat wave accelerator. We show that in general, for typical square pulse forms, the strong emission of plasma waves occur after a short transient at most of order $t_{pd} = \omega_p^{-1}(\omega_0/\omega_p)^2$. With a phase flip in the driving electromagnetic, however, we show that there are soliton solutions with the transverse and longitudinal fields strongly coupled. It is shown that the possibility of the three field soliton follows from the Hamiltonian structure of the field equations. The present work extents the wave interaction Hamiltonian (40) given earlier [11] for the classical pump depletion problem where $\hat{a}_{0,1}(x,t)$ propagate at speed $v_g = c(1-1/2\gamma^2)$ to the problem of a triple soliton structure propagating with speed $u > v_g$. The new interaction Hamiltonian clarifies the Lorentz covariance properties of the accelerating structure. Action is a Lorentz scalar and equations (43) are statements of the conservation of the action four current. The Hamiltonian (40)−(41) guarantees the total energy-momentum of the structure is infinite and uniform in extent since

$$dH/dx' = \sum_j [(\partial H/\partial I_j)\dot{I}_j + (\partial H/\partial\psi_j)\dot{\psi}_j] \equiv 0\,.$$

In the limiting case of the hyperbolic function solutions, the uniformity of the energy-momentum content is clear from the relations

$$\omega_0\hat{a}_0^2\tanh^2(qx') + \omega_1\hat{a}_1^2\operatorname{sech}^2(qx) = \omega_0\hat{a}_0^2\,.$$

Thus, it is the polarization of a phase locked, localized longitudinal electric field that propagates with speeds $\gtrsim c$ down the uniform energy-momentum structure.

5. Discussion

The integrable solution of the beat wave accelerator equations given in Sect. 4 has significance in several ways. First, it presents a clear-cut way to approach efficient conversion of laser energy to plasma wave energy and back again to transverse energy. Secondly, it has a built-in mechanism to control the soliton velocity at a desired value, including the speed of light. Thirdly, it offers a method to avoid plasma turbulence created by the wake plasma wave.

Recently McKinstrie and Dubois [16] discussed properties of triple soliton solutions and classified types A, B, and C solitons depending upon the velocity of soliton u relative to three group velocities of the constituent waves v_1, v_2 and v_3. The type A triple soliton has $u > v_1 > v_2 > v_3$, where the subscripts (1 ∼ 3) refer to the parent electromagnetic wave (tanh) with frequency ω_1, the daugther electromagnetic wave (sech) ω_2, and the daugther electrostatic wave (sech) with ω_3 respectively ($\omega_1 > \omega_2 > \omega_3 > 0$ and $\omega_1 = \omega_2 + \omega_3$). The type B soliton has $c > v_1 > v_2 > u > v_3$ and the low frequency electromagnetic wave (ω_2) has an envelope of "tanh". The type C solition has $c > v_1 > v_2 > v_3 > u$ and is not useful for acceleration.

McKinstrie et al. recommended type B ($v_3 < u < v_2 < v_1$) spatial solitary-waves for the beat wave acceleration, while Mima et al. [13] earlier suggested to use type A solitary waves with soliton velocity ($v_3 < v_2 < v_1 < u = c$). The soliton velocity $u \gtrless c$ does not mean that the energy of the three waves flow with velocity u. In fact, the total energy H and momentum P of the structure are uniform, namely, the sum of three squared amplitudes,

$$H = |A_1|^2 + |A_2|^2 + |A_3|^2$$

and the total momentum,

$$P = \frac{k_1}{\omega_1}|A_1|^2 + \frac{k_2}{\omega_2}|A_2|^2 + \frac{k_3}{\omega_3}|A_3|^2$$

are constant. Furthermore, in the type A soliton the energy in the wave A_1 has been supplied prior to the soliton propagation. Small signals of waves A_2 and A_3 are also launched in the tail of the soliton.

The small signals grow by consuming the energy of A_1, which lies ahead of it and the soliton propagates. Therefore, even if the type A soliton propagates with velocity equal or larger than c, it does not violate the special theory of relativity. A good example of similar situations is a series of linac klystrons which may be fired in sequence so as to produce a propagating longitudinal field with the phase velocity at the desired value $\gtrsim c$. In other words the triple soliton may be physically set up with either appropriate initial conditions or boundary conditions or with a mixture of initial and boundary conditions to achieve the desired speed of the accelerating structure.

The type A soliton may have a tight condition to satisfy $u > v_1 > v_2 > v_3$ if $u < c$ is required because v_1 and v_2 have to be fairly close to c. If u is close to v_1, v_2 and c, the size of the soliton becomes small. One possible way to avoid this difficulty is to introduce the plasma fiber [10] in which the condition on group velocities v_1 and v_2 are relaxed because the phase velocity v_1 and v_2 are varied by adjusting the fiber parameters.

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